LAGRANGE MULTIPLIERS

GOAL: Develop a method for constrained optimization.

Given: objective function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ and constraint functions $g_1, g_2, \dots, g_k : D \subseteq \mathbb{R}^n \to \mathbb{R}$.

Want: Optimize f subject to $g_1 = g_2 = \cdots = g_k = 0$.

↳ i.e. solutions $\vec{x} = (x_1, x_2, \dots, x_n)$ must satisfy both

① $g_1(\vec{x}) = 0, \; g_2(\vec{x}) = 0, \; \dots, \; g_k(\vec{x}) = 0$ and ☆

② $f(\vec{x})$ is absolute extreme value among all $\vec{x}$ satisfying part ①. ☆

How do we do this? Let's make a few observations:

① We can think of $g_1 = g_2 = \cdots = g_k = 0$ as giving an "admissible subset" $A \subseteq D$.

② Optimizing f then means the tangent to f coincides with the tangent to A. (See GeoGebra visualization). ☆

Because the tangent of f coincides with the tangent to the admissible subset A, we can encode this information using the gradient :)

In particular, the gradient of f is a linear combination of the gradients of the constraints at solutions:

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_K \nabla g_K .$$ ☆

where $\lambda_1, \lambda_2, \ldots, \lambda_K$ are real numbers.

NB: Trade-off : we know where to look for solutions PROVIDED we can do some algebra...

Prop (Method of Lagrange Multipliers): Suppose $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ attains an extremum subject to $g_1 = g_2 = \cdots = g_K = 0$ ☆ for $g_1, g_2, \ldots, g_K : D \subseteq \mathbb{R}^n \to \mathbb{R}$. The function given by

$$H(\vec{x}, \lambda_1, \lambda_2, \ldots, \lambda_K) = f(\vec{x}) - \lambda_1 g_1(\vec{x}) - \lambda_2 g_2(\vec{x}) - \cdots - \lambda_K g_K(\vec{x})$$ ☆

satisfies $\nabla H(\vec{x}, \vec{\lambda}) = \vec{0}$ at every solution $\vec{x}$ of the Constrained Optimization Problem for some $\vec{\lambda}$.

(NB: Formal conclusion is $\forall \vec{x}$ solution $\exists \lambda \in \mathbb{R}^K \left[\nabla H(\vec{x}, \vec{\lambda}) = \vec{0}\right] \ldots$).

Point: Solving nice Constrained Optimization Problems can be reduced to doing some algebra!

↳ The algebra can be quite hard in general... Most of the time, the equations won't be linear...

Ex: Optimize $f(x,y) = 6xy^3$ on parabola $x = y^2 - 4$.

Sol: Want to apply Lagrange Multipliers.

$\rightarrow f(x,y) = 6xy^3 \leftarrow$ objective

$g(x,y) = y^2 - x - 4$

Look at $H(x,y,\lambda) = 6xy^3 - \lambda(y^2 - x - 4)$

$$\nabla H = \langle 6y^3 + \lambda,\ 18xy^2 - 2\lambda y,\ -(y^2 - x - 4) \rangle$$

Have $\nabla H = \vec{0}$ $\iff \begin{cases} 6y^3 + \lambda = 0 \\ 18xy^2 - 2\lambda y = 0 \\ -(y^2 - x - 4) = 0 \end{cases}$

$$\iff \begin{cases} \lambda = -6y^3 \\ 18xy^2 + 12y^4 = 0 \\ y^2 - 4 = x \end{cases}$$

$$\iff \begin{cases} \lambda = -6y^3 \\ x = y^2 - 4 \\ 18(y^2-4)y^2 + 12y^4 = 0 \end{cases}$$

$$\iff \begin{cases} \lambda = -6y^3 \\ x = y^2 - 4 \\ 6y^2(3y^2 - 12 + 2y^2) = 0 \end{cases}$$

$$\iff \begin{cases} \lambda = -6y^3 \\ x = y^2 - 4 \\ 6y^2(5y^2 - 12) = 0 \end{cases}$$

$\therefore\ y = 0$ OR $y = \pm\sqrt{\frac{12}{5}} = \pm \frac{2\sqrt{3}}{5}$

Candidates for optimized solution:

$$(x,y,\lambda) \in \left\{ (-4, 0, 0),\ \left(-\frac{8}{5}, \frac{2\sqrt{3}}{5}, -6\cdot\left(\frac{2\sqrt{3}}{5}\right)^3\right),\ \left(-\frac{8}{5}, -\frac{2\sqrt{3}}{5}, +6\left(\frac{2\sqrt{3}}{5}\right)^3\right) \right\}$$

$f(-4,0) = 6 \cdot -4 \cdot 0^3 = 0$, $f\left(-\frac{8}{5}, \frac{2\sqrt{3}}{5}\right) = 6 \cdot -\frac{8}{5} \cdot \left(\frac{2\sqrt{3}}{5}\right)^3 = -6 \cdot \frac{64\sqrt{3}}{625} < 0$,

$f\left(-\frac{8}{5}, -\frac{2\sqrt{3}}{5}\right) = 6 \cdot \frac{64\sqrt{3}}{625} > 0$ $\therefore$ Extreme values occur at $\left(-\frac{8}{5}, \frac{2\sqrt{3}}{5}\right) \leftarrow$ abs min, $\left(-\frac{8}{5}, -\frac{2\sqrt{3}}{5}\right) \leftarrow$ abs max. ∎

Ex: Compute the extrema of $f(x,y) = x^2 + 2y^2$ on $x^2+y^2=1$.

Sol: Given: $f(x,y) = x^2+2y^2 \leftarrow$ objective function

$g(x,y) = x^2+y^2-1 \leftarrow$ constraint

$$H(x,y,\lambda) = (x^2+2y^2) - \lambda(x^2+y^2-1)$$

$$\nabla H = \langle 2x - 2x\lambda,\ 4y - 2y\lambda,\ -(x^2+y^2-1) \rangle$$

$$\therefore \nabla H = \vec{0} \iff \begin{cases} 2x(1-\lambda) = 0 & (1) \\ 2y(2-\lambda) = 0 & (2) \\ x^2+y^2 = 1 & (3) \end{cases}$$

So either $2x = 0$ or $\lambda = 1$.

Case 1: $2x = 0 \quad \therefore x = 0$. Thus $y^2 = 1$

So $y = \pm 1$, and $\lambda = 2$. ← need this, but doesn't affect

So candidates $(x,y) \in \{(0,1), (0,-1)\}$

Case 2: $\lambda = 1 \quad \therefore y = 0$ by (2), so by (3)

$x^2 = 1$ yields $x = \pm 1$. So candidates

$(x,y) \in \{(1,0), (-1,0)\}$.

Finally, we evaluate our objective function $f(x,y) = x^2+2y^2$

$f(0,1) = 2$, $f(0,-1) = 2$, $f(1,0) = 1$, $f(-1,0) = 1$.

$\therefore$ The maximum value is 2, and minimum value is 1 $\blacksquare$

Ex: Compute the extrema of $f(x,y,z) = e^{xyz}$ on $2x^2+y^2+z^2=24$.

Sol: $H(x,y,z,\lambda) = e^{xyz} - \lambda(2x^2+y^2+z^2-24)$

$$\nabla H = \nabla\left(e^{xyz} - \lambda(2x^2+y^2+z^2-24)\right)$$
$$= \nabla e^{xyz} - \nabla\left(\lambda(2x^2+y^2+z^2-24)\right)$$
$$= e^{xyz}\,\nabla(xyz) - \nabla\left(\lambda(2x^2+y^2+z^2-24)\right)$$
$$= e^{xyz}\langle yz, xz, xy, 0\rangle - \langle 4\lambda x, 2\lambda y, 2\lambda z, 2x^2+y^2+z^2-24\rangle$$
$$= \langle e^{xyz}\cdot yz - 4\lambda x,\ xze^{xyz} - 2\lambda y,\ xye^{xyz} - 2\lambda z,\ -(2x^2+y^2+z^2-24)\rangle$$

$\therefore \nabla H = \vec{0} \iff$
$$\begin{cases} 4\lambda x = yze^{xyz} & (1) \\ 2\lambda y = xze^{xyz} & (2) \\ 2\lambda z = xy\,e^{xyz} & (3) \\ 2x^2+y^2+z^2=24 & (4) \end{cases}$$

NB: If any of $x,y,z=0$, $f(x,y,z)=1$.

$\therefore$ We can assume $x \neq 0$, $y \neq 0$, and $z \neq 0$.

Dividing (1) by (2) yields $\frac{2x}{y} = \frac{4\lambda x}{2\lambda y} = \frac{yze^{xyz}}{xze^{xyz}} = \frac{y}{x}$

Dividing (1) by (3) yields $\frac{2x}{z} = \frac{4\lambda x}{2\lambda z} = \frac{yze^{xyz}}{xye^{xyz}} = \frac{z}{x}$

Hence we obtain $2x^2 = y^2 = z^2$. Hence (4) yields

$$24 = 2x^2+y^2+z^2 = 2x^2+2x^2+2x^2 = 6x^2$$

$\therefore x = \pm 2$. So $y = \pm 2\sqrt{2}$ and $z = \pm 2\sqrt{2}$.

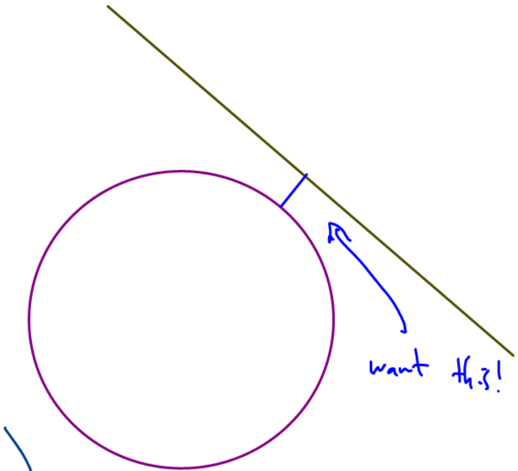
Evaluating our points $\left\{ (2, 2\sqrt{2}, 2\sqrt{2}), (2, 2\sqrt{2}, -2\sqrt{2}), (2, -2\sqrt{2}, 2\sqrt{2}), (2, -2\sqrt{2}, -2\sqrt{2}), (-2, 2\sqrt{2}, 2\sqrt{2}), (-2, 2\sqrt{2}, -2\sqrt{2}), (-2, -2\sqrt{2}, 2\sqrt{2}), (-2, -2\sqrt{2}, -2\sqrt{2}) \right\}$

$f(x,y,z) = e^{16}$ (max value) if (x,y,z) has 2 negative components and

$f(x,y,z) = e^{-16}$ (min value) otherwise.

$\boxtimes$

Ex: Find the point on the line $3x - y = 7$ closest to the circle $x^2 + y^2 = 4$.



Sol: We'll minimize the square distance. Points on the circle must be treated separately from points on the line, and we have two constraints (one for the circle, one for the line).

Problem: Minimize $f(x,y,a,b) = (x-a)^2 + (y-b)^2$ subject to

$$x^2 + y^2 = 4 \quad \text{and} \quad 3a - b = 7.$$

\+ instead of −, no effect on solution.

Define $H(x,y,a,b,\lambda,\mu) = (x-a)^2 + (y-b)^2 + \lambda(x^2+y^2-4) + \mu(3a-b-7)$.

$$\nabla H = \langle 2(x-a) + 2\lambda x,\ 2(y-b) + 2\lambda y,\ -2(x-a) + 3\mu,\ -2(y-b) - \mu,\ x^2+y^2-4,\ 3a-b-7 \rangle$$

∴ we want to solve the following system of equations:

$$\nabla H = \vec{0} \iff \begin{cases} a = x(1+\lambda) & (1) \\ b = y(1+\lambda) & (2) \\ 3\mu = 2x - 2a & (3) \\ \mu = -2y + 2b & (4) \\ x^2 + y^2 = 4 & (5) \\ 3a - b = 7 & (6) \end{cases}$$

③ and ① yield $3\mu = 2(x - x - x\lambda) = -2x\lambda \quad (7)$

Now $3a - b = 3x(1+\lambda) - y(1+\lambda) = (3x-y) + \lambda(3x-y)$ by ① and ②, but ③ and ④ yield the equation:

$$3a - b = 3 \cdot \tfrac{1}{2}(2x - 3\mu) - \tfrac{1}{2}(\mu + 2y)$$

$$= 3x - y - 5\mu = 3x - y - \tfrac{5}{3}\cdot 3\mu$$

by ⑦

$$= 3x - y - \tfrac{5}{3}(-2x\lambda)$$

Hence equating these expressions and subtracting off $3x-2y$:

$$\frac{10}{3}x\lambda = \lambda(3x-y), \text{ so } \lambda\left(\tfrac{1}{3}x + y\right) = 0.$$

Case 1: If $\lambda = 0$, then $a = x$ and $b = y$ yields $y = 3x - 7$ by ⑥, so ⑤ yields

$$x^2 + (3x-7)^2 = 4 \iff 10x^2 - 42x + 45 = 0;$$

but $(-42)^2 - 4 \cdot 10 \cdot 45 = 4(21^2 - 10\cdot 45) = 4(441 - 450) < 0$ implies this equation has no solutions. Thus, this case yields no possible solutions.

Case 2: If $\tfrac{1}{3}x + y = 0$, then ⑤ yields the equation

$$4 = x^2 + y^2 = x^2 + \left(-\tfrac{1}{3}x\right)^2 = \left(1 + \tfrac{1}{9}\right)x^2 = \tfrac{10}{9}x^2, \text{ so}$$

$$x = \pm\sqrt{\frac{4\cdot 9}{10}} = \pm\frac{6}{\sqrt{10}}, \text{ so we have}$$

$$(x,y) \in \left\{\left(\tfrac{6}{\sqrt{10}}, -\tfrac{2}{\sqrt{10}}\right), \left(-\tfrac{6}{\sqrt{10}}, \tfrac{2}{\sqrt{10}}\right)\right\}.$$

Now equations ①, ②, and ⑥ yield the following:

$$(x,y) = \left(\tfrac{6}{\sqrt{10}}, -\tfrac{2}{\sqrt{10}}\right) \Rightarrow 7 = 3a - b = (1+\lambda)(3x - y)$$

$$= (1+\lambda)\cdot\frac{20}{\sqrt{10}}$$

$$\Rightarrow \lambda = \frac{7}{20}\sqrt{10} - 1$$

and

$$(x,y) = \left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) \implies 7 = 3a - 6 = -\frac{20}{\sqrt{10}}(1+\lambda)$$

$$\implies \lambda = -\frac{7}{20}\sqrt{10} - 1$$

By equation ⑦ we see $\mu = -\frac{2}{3} x \lambda$, and ③ and ⑥ yield $a = \frac{1}{2}(2x - 3\mu)$ and $b = 3a - 7$ ←. Hence

$$(x,y) = \left(\frac{6}{\sqrt{10}}, -\frac{2}{\sqrt{10}}\right) \implies \mu = -\frac{2}{3}\cdot\frac{6}{\sqrt{10}}\left(\frac{7}{20}\sqrt{10} - 1\right) \leftarrow$$

$$\implies a = \frac{6}{\sqrt{10}} - \frac{3}{2}\cdot -\frac{2}{3}\cdot\frac{6}{\sqrt{10}}\left(\frac{7}{20}\sqrt{10} - 1\right) \leftarrow$$

$$= \frac{6}{\sqrt{10}}\cdot\left(1 + \frac{7}{20}\sqrt{10} - 1\right) = \frac{21}{10} \leftarrow$$

$$\text{and } b = 3\cdot\frac{21}{10} - 7 = -\frac{7}{10}$$

$$\implies (a,b) = \left(\frac{21}{10}, -\frac{7}{10}\right) \leftarrow$$

$$(x,y) = \left(-\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right) \implies \mu = -\frac{2}{3}\cdot -\frac{6}{\sqrt{10}}\cdot -\left(\frac{7}{20}\sqrt{10} + 1\right) \leftarrow$$

$$\implies a = -\frac{6}{\sqrt{10}} + \frac{3}{2}\cdot\frac{2}{3}\cdot\frac{6}{\sqrt{10}}\left(\frac{7}{20}\sqrt{10} + 1\right) \leftarrow$$

$$= \frac{6}{\sqrt{10}}\left(-1 + \frac{7}{20}\sqrt{10} + 1\right) = \frac{21}{10} \leftarrow$$

$$\text{and } b = 3\cdot\frac{21}{10} - 7 = -\frac{7}{10} \leftarrow$$

$$\implies (a,b) = \left(\frac{21}{10}, -\frac{7}{10}\right) \leftarrow$$

Hence the desired point on the line is $\left(\frac{21}{10}, -\frac{7}{10}\right)$ ▨

NB: Technically we would need to show that f attains a minimum. Why does it?

Ex: Find the points on the sphere $x^2+y^2+z^2=4$ closest to and furthest from $(3,1,-1)$.

Sol: We optimize square distance formula

$$f(x,y,z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

subject to constraint $g(x,y,z) = x^2+y^2+z^2-4$.

$$\nabla(f-\lambda g) = \langle 2(x-3)-2x\lambda,\ 2(y-1)-2y\lambda,\ 2(z+1)-2z\lambda,\ x^2+y^2+z^2-4\rangle$$

$$\therefore\ \nabla(f-\lambda g) = 0 \iff \begin{cases} x-3-x\lambda = 0 \\ y-1-y\lambda = 0 \\ z+1-z\lambda = 0 \\ x^2+y^2+z^2 = 4 \end{cases} \iff \begin{cases} 3 = x(1-\lambda) & (1) \\ 1 = y(1-\lambda) & (2) \\ -1 = z(1-\lambda) & (3) \\ x^2+y^2+z^2 = 4 & (4) \end{cases}$$

Now multiplying ④ by $(1-\lambda)^2$ on both sides yields the following (applying ①, ②, and ③):

$$\begin{aligned} 4(1-\lambda)^2 &= x^2(1-\lambda)^2 + y^2(1-\lambda)^2 + z^2(1-\lambda)^2 \\ &= (x(1-\lambda))^2 + (y(1-\lambda))^2 + (z(1-\lambda))^2 \\ &= 3^2 + 1^2 + (-1)^2 = 11 \end{aligned}$$

Hence $1-\lambda = \pm\frac{\sqrt{11}}{2}$, so we can solve ①, ②, and ③:

Case 1: $1-\lambda = \frac{\sqrt{11}}{2}$ yields

$$\begin{cases} x = \frac{3}{1-\lambda} = \frac{6}{\sqrt{11}} \\ y = \frac{1}{1-\lambda} = \frac{2}{\sqrt{11}} \\ z = \frac{-1}{1-\lambda} = -\frac{2}{\sqrt{11}} \end{cases}$$

$$f\left(\tfrac{6}{\sqrt{11}}, \tfrac{2}{\sqrt{11}}, -\tfrac{2}{\sqrt{11}}\right) = \left(\tfrac{6}{\sqrt{11}}-3\right)^2 + \left(\tfrac{2}{\sqrt{11}}-1\right)^2 + \left(-\tfrac{2}{\sqrt{11}}+1\right)^2$$
$$= \tfrac{36}{11} - \tfrac{36}{\sqrt{11}} + 9 + \tfrac{4}{11} - \tfrac{4}{\sqrt{11}} + 1 + \tfrac{4}{11} - \tfrac{4}{\sqrt{11}} + 1 = 15 - 4\sqrt{11}$$

Case 2: $1-\lambda = -\frac{\sqrt{11}}{2}$ yields

$$\begin{cases} x = \frac{3}{1-\lambda} = -\frac{6}{\sqrt{11}} \\ y = \frac{1}{1-\lambda} = -\frac{2}{\sqrt{11}} \\ z = \frac{-1}{1-\lambda} = \frac{2}{\sqrt{11}} \end{cases}$$

$$f\left(-\tfrac{6}{\sqrt{11}}, -\tfrac{2}{\sqrt{11}}, \tfrac{2}{\sqrt{11}}\right) = \left(-\tfrac{6}{\sqrt{11}}-3\right)^2 + \left(-\tfrac{2}{\sqrt{11}}-1\right)^2 + \left(\tfrac{2}{\sqrt{11}}+1\right)^2$$
$$= \tfrac{36}{11} + \tfrac{36}{\sqrt{11}} + 9 + 2\left(\tfrac{4}{11} + \tfrac{4}{\sqrt{11}} + 1\right) = 15 + 4\sqrt{11}$$

Hence $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ minimizes distance and $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$ maximizes distance. $\blacksquare$

Ex: What is the maximum volume of a rectangular box with an open top and surface area 12 square units?

Sol: We want to maximize $f(x,y,z) = xyz$ subject to $g(x,y,z) = xy + 2xz + 2yz - 12$. Now we compute

$$\nabla(f - \lambda g) = \nabla\left(xyz - \lambda\left(xy + 2xz + 2yz - 12\right)\right)$$

$$= \langle yz - \lambda(y+2z),\ xz - \lambda(x+2z),\ xy - 2\lambda(x+y),\ xy + 2xz + 2yz - 12 \rangle$$

$$\therefore\ \nabla(f-\lambda g) = \vec{0} \iff \begin{cases} yz = \lambda(y+2z) & (1) \\ xz = \lambda(x+2z) & (2) \\ xy = 2\lambda(x+y) & (3) \\ xy + 2xz + 2yz = 12 & (4) \end{cases}$$

Multiplying ①, ②, and ③ by x, y, and z respectively and subsequently adding yields

$$3xyz = \lambda(xy + 2xz + xy + 2yz + 2xz + 2yz)$$
$$= 2\lambda(xy + 2xz + 2yz) = 24\lambda. \quad \text{by } ④$$

Hence $f(x,y,z) = xyz = 8\lambda$ for our solution.

On the other hand, applying ①, ②, ③ to ④ yields

$$12 = xy + 2xz + 2yz$$
$$= \lambda(y + 2z + x + 2z + 2x + 2y)$$
$$= \lambda(3x + 3y + 4z), \quad \text{so } \lambda \neq 0$$

But ③ and ④ yield (together with $xyz = 8\lambda$):

$$12 = xy + 2z(x+y) = xy + \frac{xyz}{\lambda} = xy + 8, \quad \text{so } xy = 4.$$

Hence $xyz = 8\lambda$ implies $z = 2\lambda$.

Now by equation ① and $z = 2\lambda$ we obtain

$2y\lambda = yz = \lambda(y + 2z) = \lambda(y + 4\lambda)$, so we obtain

$\lambda(4\lambda - y) = 0$; thus $\lambda \neq 0$ implies $y = 4\lambda$.

By equation ② and $z = 2\lambda$ we observe

$2x\lambda = xz = \lambda(x + 2z) = \lambda(x + 4\lambda)$, so again

$\lambda(4\lambda - x) = 0$; as $\lambda \neq 0$ we see $x = 4\lambda$.

Finally, $8\lambda = xyz = (4\lambda)(4\lambda)(2\lambda) = 32\lambda^3$ yields

$$0 = 8\lambda(4\lambda^2 - 1) = 8\lambda(2\lambda + 1)(2\lambda - 1)$$

As $\lambda \neq 0$ and $\lambda = -\frac{1}{2}$ implies $x < 0$ (which violates the implicit constraints $x, y, z \geq 0$ for lengths), we obtain $\lambda = \frac{1}{2}$, and thus a maximum value

$$f(x,y,z) = 8\lambda = 8\left(\tfrac{1}{2}\right) = 4.$$ ☐

__NB__: Your textbook takes a slightly different approach to this problem... Feel free to read that!

__Ex__: Optimize $x + 2y + 3z$ subject to $x - y + z = 1$ and $x^2 + y^2 = 1$.

__Sol__: Left to you!

Ex: Compute the coordinates of the point P on the general plane $aX + by + cz = d$ closest to given point $Q = (x_0, y_0, z_0)$ in $\mathbb{R}^3$.

Sol: Minimize $f(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$

subject to $g(x,y,z) = d - ax - by - cz$.

$$\nabla(f - \lambda g) = \langle 2x - 2x_0 + a\lambda,\ 2y - 2y_0 + b\lambda,\ 2z - 2z_0 + c\lambda,\ d - ax - by - cz \rangle$$

So

$$\nabla(f - \lambda g) = 0 \iff \begin{cases} x_0 = x + \frac{a}{2}\lambda \\ y_0 = y + \frac{b}{2}\lambda \\ z_0 = z + \frac{c}{2}\lambda \\ ax + by + cz = d \end{cases}$$

Hence we obtain the following:

$$\begin{aligned} d &= ax + by + cz \\ &= a\left(x_0 - \frac{a}{2}\lambda\right) + b\left(y_0 - \frac{b}{2}\lambda\right) + c\left(z_0 - \frac{c}{2}\lambda\right) \\ &= ax_0 + by_0 + cz_0 - \lambda\left(\frac{a^2 + b^2 + c^2}{2}\right) \end{aligned}$$

Hence solving yields $\lambda = \dfrac{2(ax_0 + by_0 + cz_0)}{a^2 + b^2 + c^2}$.

Thus the minimum distance is obtained at point

$$\begin{aligned} (x,y,z) &= \left(x_0 - \frac{\lambda}{2}a,\ y_0 - \frac{\lambda}{2}b,\ z_0 - \frac{\lambda}{2}c\right) \\ &= \left(x_0 - \frac{a(ax_0 + by_0 + cz_0)}{a^2 + b^2 + c^2},\ y_0 - \frac{b(ax_0 + by_0 + cz_0)}{a^2 + b^2 + c^2},\ z_0 - \frac{c(ax_0 + by_0 + cz_0)}{a^2 + b^2 + c^2}\right) \end{aligned}$$

$\boxtimes$